2224 – 7101M

Markscheme

May 2024

Mathematics: analysis and approaches

Higher level

Paper 1

33 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

No calculator is allowed. The use of any calculator on this paper is malpractice and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

Section A

1. (a) attempt to form equation for the sum of frequencies $=16$ or mean $=3$ ***(M1)***

$p+q+4+2+3=16 \left(\Rightarrow p+q=7\right)$ ***A1***

$\frac{p+2q+12+8+18}{16}=3\left(\Rightarrow p+2q=10\right)$ OR $\frac{p+2q+12+8+18}{9+p+q}=3\left(\Rightarrow 2p+q=11\right)$ ***A1***

attempt to eliminate one variable from their equations ***(M1)***

$p+2\left(7-p\right)+38=48$ OR $2\left(7-q\right)+q=11$

$p=4$ and $q=3$ ***A1***

Note: Award ***M1A0A0M0A1*** for $p=4, q=3$ with no working.

[5 marks]

(b) mean final score $=30$ ***A1***

[1 mark]

Total [6 marks]

2. (a) $\log_{10}1-\log_{10}a$ OR $\log_{10}a^{-1}=-\log_{10}a$ OR $\log_{10}10^{-\frac{1}{3}}$ OR $10^x=\frac{1}{10^{\frac{1}{3}}}$ ***(A1)***

$=-\frac{1}{3}$ ***A1***

[2 marks]

(b) $\frac{\log_{10}a}{\log_{10}1000}$ OR $\frac{1}{3}\log_{1000}10$ OR $\log_{1000}\sqrt[3]{1000^{\frac{1}{3}}}$ OR $10^{\frac{1}{3}}=1000^x\left(=\left(10^3\right)^x\right)$ ***(A1)***

$\frac{\log_{10}a}{3}$ OR $\frac{1}{3}\log_{1000}1000^{\frac{1}{3}}$ OR $\log_{1000}1000^{\frac{1}{9}}$ OR $3x=\frac{1}{3}$ ***(A1)***

$=\frac{1}{9}$ ***A1***

[3 marks]

Total [5 marks]

3. (a) $2r + r\theta = 10$ ***A1***

$\frac{1}{2}r^2\theta = 6.25$ ***A1***

attempt to eliminate θ to obtain an equation in r ***M1***

correct intermediate equation in r ***A1***

$10 - 2r = \frac{25}{2r}$ OR $\frac{10}{r} - 2 = \frac{25}{2r^2}$ OR $\frac{1}{2}r^2\left(\frac{10}{r} - 2\right) = 6.25$ OR $12.5 + 2r^2 = 10r$

$4r^2 - 20r + 25 = 0$ ***AG***

[4 marks]

(b) attempt to solve quadratic by factorizing or use of formula or completing the square ***(M1)***

$(2r-5)^2 = 0$ OR $r = \frac{20 \pm \sqrt{(-20)^2 - 4(4)(25)}}{2(4)}\left(= \frac{20 \pm \sqrt{400-400}}{8}\right)$

$r = \frac{5}{2}$ ***A1***

attempt to substitute their value of r into their perimeter or area equation ***(M1)***

$\theta = \frac{10 - 2\left(\frac{5}{2}\right)}{\left(\frac{5}{2}\right)}$ or $\theta = \frac{25}{2\left(\frac{5}{2}\right)^2}$

$\theta = 2$ ***A1***

[4 marks]

Total [8 marks]

4. (a) recognising $\cos x = 2\sin x\cos x$ ***(M1)***

$(\cos x \neq 0)$ so $\sin x = \frac{1}{2}$ OR one correct value (accept degrees) ***(A1)***

x - coordinates $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ ***A1***

Note: Award ***(M1)(A1)A0*** for solutions of 30° and 150°.

[3 marks]

(b) **METHOD 1**

attempt to integrate $\pm(\cos x - \sin 2x)$ ***(M1)***

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - \sin 2x)\,\mathrm{d}x$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (\cos x - 2\sin x\cos x)\,\mathrm{d}x$

$= \left[\sin x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ OR $= \left[\sin x - \sin^2 x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for $\pm$correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract ***M1***

$= \left(\sin\left(\frac{5\pi}{6}\right) + \frac{1}{2}\cos\left(\frac{5\pi}{3}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos(\pi)\right)$ OR

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)\right) - \left(\sin\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right)\right)$

$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(1 - \frac{1}{2}\right)$ OR $= \left(\frac{1}{2} - \frac{1}{4}\right) - (1 - 1)$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B. Accept work done in degrees.

continued…

Question 4 continued

METHOD 2

$\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \mathrm{d}x = [\sin x]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ and $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \mathrm{d}x = \left[-\frac{1}{2}\cos 2x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}$ ***A1***

Note: Award ***A1*** for correct integration. Condone incorrect or absent limits up to this point.

attempt to substitute their limits into their integral and subtract (for both integrals) ***M1***

$\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)$ and $-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)$

attempt to subtract the two integrals in either order (seen anywhere) ***(M1)***

$\left(\sin\left(\frac{5\pi}{6}\right) - \sin\left(\frac{\pi}{2}\right)\right) - \left(-\frac{1}{2}\cos\left(\frac{5\pi}{3}\right) + \frac{1}{2}\cos(\pi)\right)$ OR $\int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \cos x \mathrm{d}x - \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} \sin 2x \mathrm{d}x$

$= \left(\frac{1}{2} - 1\right) - \left(\frac{1}{4} - \frac{1}{2}\right) \quad \left(= -\frac{1}{4}\right)$

area $= \frac{1}{4}$ ***A1***

Note: Award all corresponding marks as appropriate for finding the area between A and B. Accept work done in degrees.

[4 marks]

Total [7 marks]

5. (a) $S_n = \dfrac{10^n - 1}{9}$ ***A1***

$(a = 10, b = 9)$

[1 mark]

(b) **METHOD 1**

$S_1 + S_2 + S_3 + \ldots + S_n$

$= \dfrac{10-1}{9} + \dfrac{10^2-1}{9} + \ldots + \dfrac{10^n-1}{9}$ ***(A1)***

$= \dfrac{10-1+10^2-1+10^3-1+\ldots+10^n-1}{9}$ OR $\dfrac{9\left(10-1+10^2-1+10^3-1+\ldots+10^n-1\right)}{81}$

attempt to use geometric series formula on powers of 10, and collect -1's together ***M1***

$10+10^2+10^3+\ldots+10^n = \dfrac{10\left(10^n-1\right)}{10-1}$ and $-1-1-1\ldots = -n$ ***A1***

$= \dfrac{\dfrac{10\left(10^n-1\right)}{10-1} - n}{9}$ OR $\dfrac{9\left(\dfrac{10\left(10^n-1\right)}{10-1}\right) - 9n}{81}$ ***A1***

Note: Award ***A1*** for any correct intermediate expression.

$= \dfrac{10\left(10^n-1\right) - 9n}{81}$ ***AG***

continued…

Question 5 continued

METHOD 2

attempt to create sum using sigma notation with S_n ***M1***

$$\sum_{i=1}^{n}\frac{10^i-1}{9} \quad \left(=\frac{1}{9}\left(\sum_{i=1}^{n}10^i-\sum_{i=1}^{n}1\right)\right)$$

$$\sum_{i=1}^{n}10^i=\frac{10\left(10^n-1\right)}{9}$$ ***A1***

$$\sum_{i=1}^{n}1=n$$ ***A1***

$$=\frac{1}{9}\left(\frac{10\left(10^n-1\right)}{9}-n\right) \text{ OR } \frac{1}{9}\left(\frac{10\left(10^n-1\right)-9n}{9}\right)$$ ***A1***

$$=\frac{10\left(10^n-1\right)-9n}{81}$$ ***AG***

continued…

Question 5 continued

METHOD 3

let $\mathrm{P}(n)$ be the proposition that $S_1 + S_2 + S_3 + ... + S_n = \frac{10\left(10^n - 1\right) - 9n}{81}$

considering $\mathrm{P}(1)$:

$\mathrm{LHS} = S_1 = \frac{10^1 - 1}{9} = 1$ and $\mathrm{RHS} = \frac{10\left(10^1 - 1\right) - 9(1)}{81} = 1$ and so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *i.e.* $S_1 + S_2 + S_3 + ... + S_k = \frac{10\left(10^k - 1\right) - 9k}{81}$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$" or "$n = k$ is true". Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$S_1 + S_2 + S_3 + ... + S_{k+1} = \frac{10\left(10^k - 1\right) - 9k}{81} + \frac{10^{k+1} - 1}{9}$

$= \frac{10^{k+1} - 10 - 9k + 9\left(10^{k+1}\right) - 9}{81}$ ***A1***

$= \frac{10\left(10^{k+1} - 1\right) - 9(k+1)}{81}$

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true ***R1***

(for all integers $n \geq 1$)

Note: To obtain the final ***R1***, the first ***R1*** and ***A1*** must have been awarded.

[4 marks]

Total [5 marks]

6. **METHOD 1**

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1.***

$\pi\int_0^{\frac{\sqrt{\pi}}{2}}\left(f(x)\right)^2\mathrm{d}x$

$\pi\int_0^{\frac{\sqrt{\pi}}{2}}x\sin\left(x^2\right)\mathrm{d}x$ ***A1***

EITHER

attempt to use integration by substitution ***M1***

$\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\sin(u)\mathrm{d}u$

Note: Award ***M1*** for $u=x^2\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2x$

$=\left[-\frac{\pi}{2}\cos(u)\right]_0^{\frac{\pi}{4}}$ ***A1***

OR

attempt to integrate by inspection ***(M1)***

$\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}}2x\sin\left(x^2\right)\mathrm{d}x$ OR $\frac{\pi}{2}\int_0^{\frac{\sqrt{\pi}}{2}}\sin\left(x^2\right)\mathrm{d}\left(x^2\right)$

$=\left[-\frac{\pi}{2}\cos\left(x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$ ***A1***

Note: Condone incorrect or absent limits for ***M1***. The correct limits may be seen or implied by later work for the ***A1***.

THEN

$=\left(-\frac{\pi}{2}\cos\left(\frac{\pi}{4}\right)\right)-\left(-\frac{\pi}{2}\cos(0)\right)$ (or equivalent) ***(A1)***

$=-\frac{\pi}{2\sqrt{2}}+\frac{\pi}{2}$ OR $-\frac{\pi\sqrt{2}}{4}+\frac{\pi}{2}$ OR $\frac{\pi}{2}\left(-\frac{1}{\sqrt{2}}+1\right)$ OR $\frac{\pi}{2}\left(-\frac{\sqrt{2}}{2}+1\right)$ ***A1***

$=\frac{\pi\left(2-\sqrt{2}\right)}{4}$ ***AG***

continued…

Question 6 continued

METHOD 2

attempt to find an integral involving π and the square of $f(x)$ ***M1***

Note: Condone incorrect or absent limits for this ***M1***.

$\pi\int_0^{\frac{\sqrt{\pi}}{2}} \left(f(x)\right)^2 \mathrm{d}x$

$\pi\int_0^{\frac{\sqrt{\pi}}{2}} x\sin\left(x^2\right)\mathrm{d}x$ ***A1***

attempt to use integration by substitution ***M1***

$u = \cos\left(x^2\right) \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2x\sin\left(x^2\right)$

Note: Award ***M1*** for $u = \cos\left(x^2\right)$

$= -\frac{\pi}{2}\int_{-\frac{1}{\sqrt{2}}}^{-1} \mathrm{d}u$

$= \left[-\frac{\pi}{2}u\right]_{-\frac{1}{\sqrt{2}}}^{-1}$ (or equivalent) ***A1A1***

Note: Condone incorrect or absent limits for ***M1***. ***A1*** for $-\frac{\pi}{2}u$ and ***A1*** for both correct limits.

$= \frac{\pi}{2} - \frac{\pi}{2\sqrt{2}}$ OR $\frac{\pi}{2} - \frac{\pi\sqrt{2}}{4}$ OR $\frac{\pi}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$ OR $\frac{\pi}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$ ***A1***

$= \frac{\pi\left(2-\sqrt{2}\right)}{4}$ ***AG***

Total [6 marks]

7. Base case $n=1$: LHS $= {}^{1}C_1 = 1$ and RHS $= {}^{2}C_2 = 1$, so true for $n=1$ ***R1***

> **Note:** Award ***R0*** if the value of ${}^{1}C_1$ and ${}^{2}C_2$ are not evaluated.
> Subsequent marks can still be awarded.

assume true for $n=k$ ie $\sum_{r=1}^{k} {}^{r}C_1 = {}^{k+1}C_2$ for some $k \in \mathbb{Z}^+$ ***M1***

> **Note:** The assumption of truth must be clear.
> Award ***M0*** for statements such as “let $n=k$” or “$n=k$ is true”.
> Subsequent marks can still be awarded.

consider $n=k+1$

LHS $= \sum_{r=1}^{k+1} {}^{r}C_1$

$= \sum_{r=1}^{k} {}^{r}C_1 + {}^{k+1}C_1$ ***(M1)***

$= {}^{k+1}C_2 + {}^{k+1}C_1$ OR $\frac{(k+1)!}{2(k-1)!} + \frac{(k+1)!}{k!}$ ***A1***

EITHER

attempt to cancel factorials and use a common denominator ***M1***

$= \frac{(k+1)k + 2(k+1)}{2} \left(= \frac{(k+2)(k+1)}{2} \right)$

OR

attempt to use a common denominator ***M1***

$= \frac{k(k+1)!}{2k!} + \frac{2(k+1)!}{2k!} \left(= \frac{(k+2)(k+1)!}{2k!} \right)$

THEN

$= \frac{(k+2)!}{2!k!} \left(= \frac{(k+2)!}{2!(k+2-2)!} \right)$ ***A1***

$= {}^{k+2}C_2$

since true for $n=1$, and true for $n=k$ implies true for $n=k+1$,

therefore true for all $n \in \mathbb{Z}^+$ ***R1***

> **Note:** Only award the final ***R1*** if 4 of the previous 6 marks have been awarded.

Total *[7 marks]*

Note: Throughout this question, condone presence of any additional terms once the first two correct terms are seen.

8. (a) (i) $\sin\left(x^2\right) = x^2 - \frac{\left(x^2\right)^3}{3!} + \ldots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \ldots \left(= x^2 - \frac{x^6}{6} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

(ii) **METHOD 1**

attempt to square their series for $\sin\left(x^2\right)$ ***(M1)***

$\left(\sin\left(x^2\right)\right)^2 = \left(x^2 - \frac{x^6}{3!} + \ldots\right)^2$

Note: Award ***M0*** for $\left(x^2\right)^2 - \left(\frac{x^6}{3!}\right)^2 + \ldots$.

$= x^4 - \frac{2x^8}{3!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

METHOD 2

attempt to use the identity $\sin^2\left(x^2\right) = \frac{1-\cos\left(2x^2\right)}{2}$ ***(M1)***

$\sin^2\left(x^2\right) = \frac{1}{2}\left(1 - \left(1 - \frac{\left(2x^2\right)^2}{2!} + \frac{\left(2x^2\right)^4}{4!}\right)\right)$

$= x^4 - \frac{8x^8}{4!} + \ldots \left(= x^4 - \frac{x^8}{3} + \ldots\right)$ ***A1A1***

Note: Award ***A1*** for each term.

[5 marks]

continued…

Question 8 continued

(b) **METHOD 1**

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=\dfrac{\mathrm{d}\left(\left(\sin\left(x^2\right)\right)^2\right)}{\mathrm{d}x}$ ***(M1)***

$=4x^3-\dfrac{8x^7}{3}+\ldots$ ***A1***

METHOD 2

recognition that $4x\sin\left(x^2\right)\cos\left(x^2\right)=2x\sin\left(2x^2\right)$ ***(M1)***

$=2x\left(2x^2-\dfrac{\left(2x^2\right)^3}{3!}+\ldots\right)$

$=4x^3-\dfrac{8x^7}{3}+\ldots$ ***A1***

METHOD 3

$4x\sin\left(x^2\right)\cos\left(x^2\right)$

$=4x\left(x^2-\dfrac{x^6}{3!}+\ldots\right)\left(1-\dfrac{x^4}{2!}+\ldots\right)$ ***(A1)***

$=4x^3-\dfrac{8x^7}{3}+\ldots$ ***A1***

continued…

Question 8 continued

METHOD 4

recognition that $2x\cos\left(x^2\right)=\dfrac{\mathrm{d}\left(\sin\left(x^2\right)\right)}{\mathrm{d}x}$ ***(M1)***

$4x\sin\left(x^2\right)\cos\left(x^2\right)$

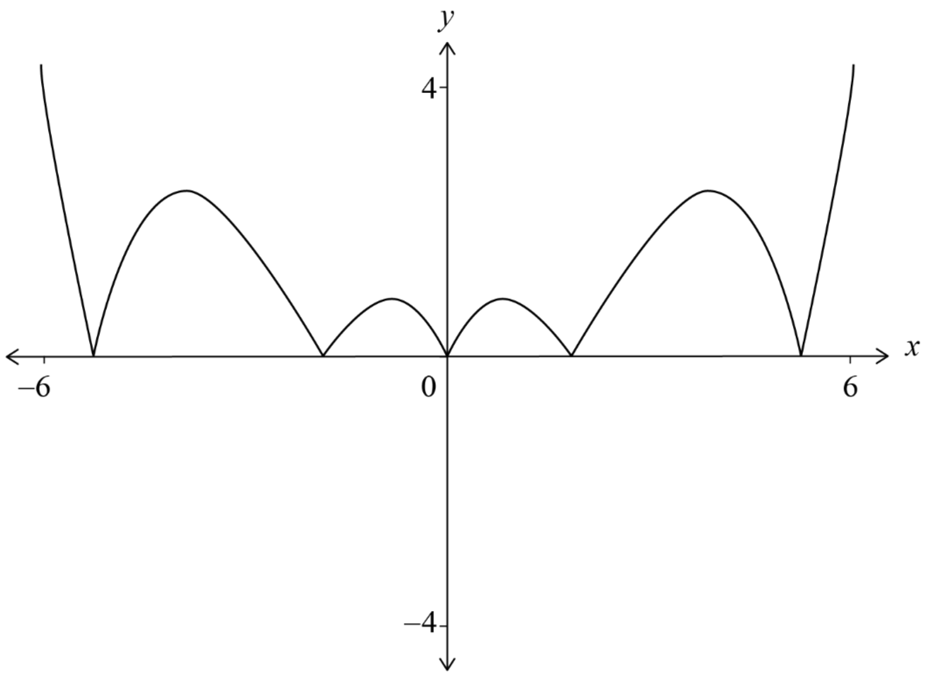
$=2\left(x^2-\dfrac{x^6}{3!}+\ldots\right)\left(2x-\dfrac{6x^5}{2!}+\ldots\right)$

$=4x^3-\dfrac{8x^7}{3}+\ldots$ ***A1***

[2 marks]

Total [7 marks]

9. (a)



reflection of all negative sections in x-axis ***(M1)***

approximately correct graph with sharp points (cusps) at x-intercepts ***A1***

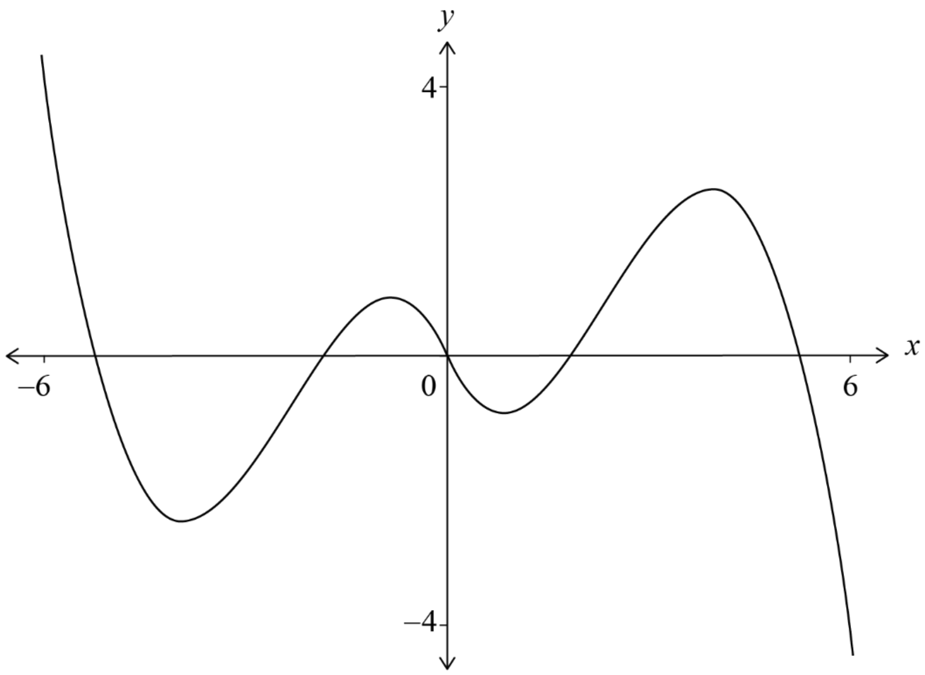
Note:	Award ***A1*** only if the heights of the maximum in the middle are lower than the heights of the maximum at the ends.

[2 marks]

continued…

Question 9 continued

(b)



A1A1

Note: Award ***A1*** for right hand side unchanged and ***A1*** for rotation 180° about the origin.

[2 marks]

(c) (i) -1.6 ***A1***

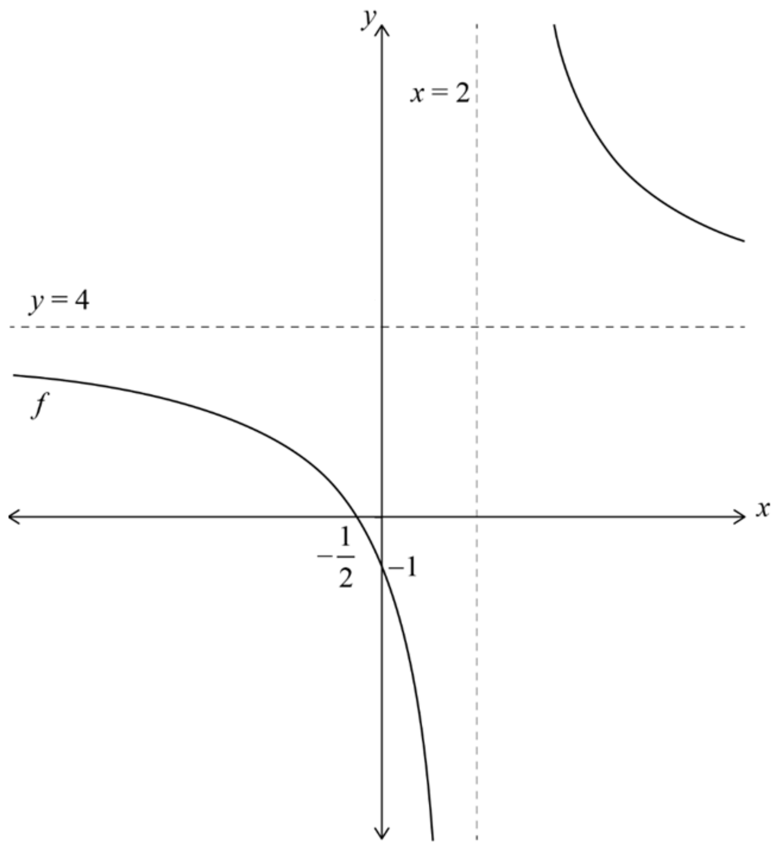
(ii) 3.2 ***A1***

[2 marks]

Total [6 marks]

Section B

10. (a)



vertical asymptote $x = 2$ sketched and labelled with correct equation ***A1***

horizontal asymptote $y = 4$ sketched and labelled with correct equation ***A1***

For an approximate rational function shape:

labelled intercepts $-\frac{1}{2}$ on x-axis, -1 on y-axis ***A1A1***

two branches in correct opposite quadrants with correct asymptotic behaviour ***A1***

Note:	These marks may be awarded independently.

[5 marks]

continued…

Question 10 continued

(b) $y \neq 4$ (or equivalent) ***A1***

[1 mark]

(c) $2+\frac{5}{2}$ OR $\left(-\frac{1}{2}\right)+2\times\frac{5}{2}$ OR $\frac{-\frac{1}{2}+p}{2}=2$ OR $-4=-p+\frac{1}{2}$ ***A1***

$p=\frac{9}{2}$ ***AG***

[1 mark]

(d) **METHOD 1**

attempt to substitute both roots to form a quadratic ***(M1)***

EITHER

$\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $x^2-\left(-\frac{1}{2}+\frac{9}{2}\right)x+\left(-\frac{1}{2}\times\frac{9}{2}\right)$

$=x^2-4x-\frac{9}{4}$ ***A1A1***

$\left(b=-4, c=-\frac{9}{4}\right)$

Note: Award ***A1*** for each correct value. They may be embedded or stated explicitly.

OR

$(2x+1)(2x-9)=4\left(x^2-4x-\frac{9}{4}\right)$

$b=-4, c=-\frac{9}{4}$ ***A1A1***

Note: Award ***A1*** for each correct value. They must be stated explicitly.

continued…

Question 10 continued

METHOD 2

$-\frac{b}{2}=2$ OR $4+b=0 \Rightarrow b=-4$ ***A1***

attempt to form a valid equation to find c using their b ***(M1)***

$\left(-\frac{1}{2}\right)^2+-4\left(-\frac{1}{2}\right)+c=0$ OR $\left(\frac{9}{2}\right)^2+-4\left(\frac{9}{2}\right)+c=0$

$c=-\frac{9}{4}$ ***A1***

METHOD 3

attempt to form two valid equations in b and c ***(M1)***

$\left(-\frac{1}{2}\right)^2+b\left(-\frac{1}{2}\right)+c=0$, $\left(\frac{9}{2}\right)^2+b\left(\frac{9}{2}\right)+c=0$

$b=-4, c=-\frac{9}{4}$ ***A1A1***

METHOD 4

attempt to write $g(x)$ in the form $(x-h)^2+k$ and substitute for x, h and $g(x)$ ***(M1)***

$\left(-\frac{1}{2}-2\right)^2+k=0 \Rightarrow k=-\frac{25}{4}$

$(x-2)^2-\frac{25}{4}$

$=x^2-4x-\frac{9}{4}$ ***A1A1***

$\left(b=-4, c=-\frac{9}{4}\right)$

Note: Award ***A1*** for each correct value. They may be embedded or stated explicitly.

[3 marks]

continued…

Question 10 continued

(e) attempt to substitute $x = 2$ into their $g(x)$ OR

complete the square on their $g(x)$ (may be seen in part (d)) ***(M1)***

$y = -\frac{25}{4}$ ***A1***

[2 marks]

(f) $\frac{4x+2}{x-2} = \left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $\frac{4x+2}{x-2} = x^2 - 4x - \frac{9}{4}$

attempt to form a cubic equation ***(M1)***

EITHER

$4x+2 = (x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$ OR $4x+2 = \left(x^2-4x-\frac{9}{4}\right)(x-2)$ OR

$(x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)-4x-2$ OR $(x-2)\left(x^2-4x-\frac{9}{4}\right)-4x-2$

$x^3 + \ldots + \frac{5}{2}(=0)$ OR $4x^3 + \ldots + 10(=0)$ ***(A1)(A1)***

Note: Award ***(A1)*** for each of the terms x^3 and $\frac{5}{2}$ or $4x^3$ and 10. Ignore extra terms.

product of roots = $\left(\frac{(-1)^3 \times \frac{5}{2}}{1}\right)$ OR $\left(\frac{(-1)^3 \times 10}{4}\right)$

$= -\frac{5}{2}$ ***A1***

continued…

Question 10 continued

OR

$4\left(x+\frac{1}{2}\right)=(x-2)\left(x+\frac{1}{2}\right)\left(x-\frac{9}{2}\right)$

$x=-\frac{1}{2}$ ***(A1)***

or $4=x^2+\ldots+9 \Rightarrow x^2+\ldots+5=0$

product of roots of quadratic is 5 ***(A1)***

product is therefore $-\frac{1}{2}\times 5$

$=-\frac{5}{2}$ ***A1***

[4 marks]

Total [16 marks]

11. (a) attempt to substitute -1 into $P(x)$ OR use of synthetic division OR long division ***M1***

$3(-1)^3+5(-1)^2+(-1)-1=0$ OR

	3	5	1	-1
-1		-3	-2	1
	3	2	-1	0

OR
$$\begin{array}{r} 3x^2+2x-1 \\ x+1\overline{)3x^3+5x^2+x-1} \\ \underline{3x^3+3x^2} \\ 2x^2+x \\ \underline{2x^2+2x} \\ -x-1 \\ \underline{-x-1} \\ 0 \end{array}$$
A1

[2 marks]

(b) attempt to divide $P(x)$ by $(x+1)$ e.g. using long division or synthetic division ***(M1)***

$P(x)=(x+1)(3x^2+2x-1)$ ***(A1)***

$=(x+1)(x+1)(3x-1)\left(=(x+1)^2(3x-1)\right)$ ***A1***

[3 marks]

(c) $\dfrac{1}{(x+1)(2x+1)}\equiv\dfrac{A}{x+1}+\dfrac{B}{2x+1}\Rightarrow 1\equiv A(2x+1)+B(x+1)$

attempt to equate both coefficients OR substitute two values eg -1 and $-\dfrac{1}{2}$ ***(M1)***

$2A+B=0$ and $A+B=1$ OR $1=-A$ and $1=\dfrac{1}{2}B$

$A=-1$ and $B=2$ ***A1A1***

Note: Award ***A1*** for each value.

$$\frac{1}{(x+1)(2x+1)}=-\frac{1}{x+1}+\frac{2}{2x+1}$$

[3 marks]

continued…

Question 11 continued

(d) $\dfrac{1}{(x+1)(x+1)(2x+1)}$

$= \dfrac{1}{(x+1)}\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)$ ***(A1)***

$= -\dfrac{1}{(x+1)^2}+\dfrac{2}{(2x+1)(x+1)}\left(= -\dfrac{1}{(x+1)^2}+2\left(-\dfrac{1}{x+1}+\dfrac{2}{2x+1}\right)\right)$ ***A1***

$= \dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}$ ***AG***

Note: Award ***A1A0*** for follow through from incorrect values in part (c).

[2 marks]

(e) attempt to integrate at least one term in $\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)$ ***(M1)***

$\int\left(\dfrac{4}{2x+1}-\dfrac{2}{x+1}-\dfrac{1}{(x+1)^2}\right)\mathrm{d}x$

$= 2\ln|2x+1|-2\ln|x+1|+\dfrac{1}{x+1}(+c)$ ***A1A1A1***

Note: Award ***A1*** for each correct term. Award a maximum of ***M1A1A0A1*** if modulus signs are omitted. Condone the absence of $+c$.

[4 marks]

(f) (i) **METHOD 1**

attempt to cancel factors and substitute $x=-1$ ***(M1)***

$\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(\dfrac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right) = \lim_{x\to -1}\left(\dfrac{3x-1}{2x+1}\right) = \dfrac{3(-1)-1}{2(-1)+1}$

$= 4$ ***A1***

continued…

Question 11 continued

METHOD 2

attempt to expand denominator, differentiate numerator and denominator twice and substitute $x = -1$ ***(M1)***

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right) = \lim_{x\to -1}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right) = \lim_{x\to -1}\left(\frac{18x+10}{12x+10}\right) = \frac{18(-1)+10}{12(-1)+10}$$

$= 4$ ***A1***

(ii) **METHOD 1**

attempt to consider coefficients of x^3 or divide all terms by x^3

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{3x^3+\ldots}{2x^3+\ldots}\right) \text{ or } \lim_{x\to\infty}\left(\frac{3+\text{terms which tend to 0}}{2+\text{terms which tend to 0}}\right)$$

$= \frac{3}{2}$ ***A1***

METHOD 2

attempt to cancel factors and consider coefficients of x or divide all terms by x

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{(x+1)^2(3x-1)}{(x+1)^2(2x+1)}\right) = \lim_{x\to\infty}\left(\frac{3x-1}{2x+1}\right) \text{ or } \lim_{x\to\infty}\left(\frac{3-\frac{1}{x}}{2+\frac{1}{x}}\right)$$

$= \frac{3}{2}$ ***A1***

METHOD 3

attempt to expand denominator, differentiate numerator and denominator three times

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{3x^3+5x^2+x-1}{2x^3+5x^2+4x+1}\right) = \lim_{x\to\infty}\left(\frac{9x^2+10x+1}{6x^2+10x+4}\right) = \lim_{x\to\infty}\left(\frac{18x+10}{12x+10}\right) = \lim_{x\to\infty}\left(\frac{18}{12}\right)$$

$= \frac{3}{2}$ ***A1***

Note: If the ***M1*** has not been awarded in part (i) it can be awarded in part (ii).

[3 marks]

Total [17 marks]

12. (a) attempt to expand the brackets or attempt to find modulus and argument of ϕ ***(M1)***

$(a+b\text{i})^3 = a^3 + 3a^2b\text{i} + 3a(b\text{i})^2 + (b\text{i})^3$ OR $\left(\sqrt{a^2+b^2}\right)^3 \operatorname{cis}\left(3\arctan\left(\frac{b}{a}\right)\right)$

(i) real part is $a^3 - 3ab^2$ OR $(a^2+b^2)^{\frac{3}{2}} \cos\left(3\arctan\left(\frac{b}{a}\right)\right)$ ***A1***

(ii) imaginary part is $3a^2b - b^3$ OR $(a^2+b^2)^{\frac{3}{2}} \sin\left(3\arctan\left(\frac{b}{a}\right)\right)$ ***A1***

Note: Award ***(M1)A1A0*** for $(a^3-3ab^2)+(3a^2b-b^3)\text{i}$ OR
$(a^2+b^2)^{\frac{3}{2}} \cos\left(3\arctan\left(\frac{b}{a}\right)\right) + (a^2+b^2)^{\frac{3}{2}} \text{i}\sin\left(3\arctan\left(\frac{b}{a}\right)\right)$

For (ii) condone $(3a^2b-b^3)\text{i}$ OR $(a^2+b^2)^{\frac{3}{2}} \text{i}\sin\left(3\arctan\left(\frac{b}{a}\right)\right)$

[3 marks]

(b) attempt to substitute $a=1$ and $b=\sqrt{3}$ into their real or imaginary part found in (a)
OR to expand the brackets OR to use polar form ***M1***

$(1-9)+(3\sqrt{3}-3\sqrt{3})\text{i}$ OR $(2\sqrt{3}\text{i}-2)(1+\sqrt{3}\text{i}) = 2\sqrt{3}\text{i}-2-6-2\sqrt{3}\text{i}$ OR

$\left(2\text{e}^{\frac{\text{i}\pi}{3}}\right)^3 = 8\text{e}^{\text{i}\pi}$ OR $(2\operatorname{cis}(60^\circ))^3 = 8\operatorname{cis}(180^\circ)$ ***A1***

$=-8$ ***AG***

[2 marks]

(c) $v=-2,\ w=1-\sqrt{3}\text{i}$ ***A1A1***

Note: Award ***A1A0*** for $v=1-\sqrt{3}\text{i},\ w=-2$ or if the labels v and w are not clearly specified or missing. Candidates may be awarded full ***FT*** marks for subsequent parts.

[2 marks]

continued…

Question 12 continued

(d) **METHOD 1**

triangle UVW has height $h = 3$ and base $b = 2\sqrt{3}$ ***(A1)***

attempt to find area of triangle with their height and base ***(M1)***

area = $\frac{1}{2} \times 2\sqrt{3} \times 3$

$= 3\sqrt{3}$ (square units) ***A1***

METHOD 2

triangle UVW has sides of length $\left(\sqrt{3^2 + \left(\sqrt{3}\right)^2} =\right)\sqrt{12}$ ***(A1)***

attempt to find area of equilateral triangle with their side length ***(M1)***

area = $\frac{1}{2}\left(\sqrt{12}\right)^2 \sin\frac{\pi}{3}$ OR $\frac{1}{2}\sqrt{12}(3)$ OR $\left(\sqrt{12}\right)^2 \times \frac{\sqrt{3}}{4}$

$= 3\sqrt{3}$ (square units) ***A1***

METHOD 3

triangle UVO has sides of length $\left(\sqrt{1^2 + \left(\sqrt{3}\right)^2} =\right)2$ ***(A1)***

attempt to find area of three isosceles triangles with their side length and angle $\frac{2\pi}{3}$ ***(M1)***

area $= 3\left(\frac{1}{2}(2)^2 \sin\frac{2\pi}{3}\right)$

$= 3\sqrt{3}$ (square units) ***A1***

[3 marks]

continued…

Question 12 continued

(e) attempt to express u, v or w in the form $r\mathrm{e}^{\mathrm{i}\theta}$ and multiply by $\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}$ ***(M1)***

$\left(u' = 2\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}} =\right) 2\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}}$ ***A1***

$\left(v' = 2\mathrm{e}^{\pi\mathrm{i}}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}} = 2\mathrm{e}^{\frac{5\pi}{4}\mathrm{i}} =\right) 2\mathrm{e}^{-\frac{3\pi}{4}\mathrm{i}}$ ***A1***

$\left(w' = 2\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}\mathrm{e}^{\frac{\pi}{4}\mathrm{i}} =\right) 2\mathrm{e}^{-\frac{\pi}{12}\mathrm{i}}$ ***A1***

Note: These ***A1*** marks should be awarded independently and in any order.

[4 marks]

(f) **EITHER**

attempt to find one of $(u')^3, (v')^3$ or $(w')^3$ ***(M1)***

$(u')^3 = \left(2\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}}\right)^3 = 8\mathrm{e}^{\frac{7\pi}{4}\mathrm{i}} = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ OR $(v')^3 = \left(2\mathrm{e}^{-\frac{3\pi}{4}\mathrm{i}}\right)^3 = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ OR

$(w')^3 = \left(2\mathrm{e}^{-\frac{\pi}{12}\mathrm{i}}\right)^3 = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ ***(A1)***

OR

attempt to find product of their three roots u', v' and w' ***(M1)***

$u' \times v' \times w' = (c + d\,\mathrm{i})$

$2\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}} \times 2\mathrm{e}^{-\frac{3\pi}{4}\mathrm{i}} \times 2\mathrm{e}^{-\frac{\pi}{12}\mathrm{i}} = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ OR $2\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}} \times 2\mathrm{e}^{\frac{5\pi}{4}\mathrm{i}} \times 2\mathrm{e}^{-\frac{\pi}{12}\mathrm{i}} = 8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ (or equivalent) ***(A1)***

OR

attempt to find $\left(z\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\right)^3$ for any z such that $z^3 = -8$ OR to rotate -8 by $\frac{3\pi}{4}$ ***(M1)***

$\left(\left(z\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}\right)^3 = z^3\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}} =\right) -8\mathrm{e}^{\frac{3\pi}{4}\mathrm{i}}$ OR $8\mathrm{e}^{-\frac{\pi}{4}\mathrm{i}}$ OR $8\,\mathrm{cis}\left(-45^\circ\right)$ ***(A1)***

continued…

Question 12 continued

THEN

$8e^{-\frac{\pi}{4}i} = 8\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{8}{\sqrt{2}}(1-i)$

$= 4\sqrt{2} - 4\sqrt{2}i$ ***A1***

Note: Accept $c = \frac{8}{\sqrt{2}}, d = -\frac{8}{\sqrt{2}}$.

[3 marks]

(g) **METHOD 1**

attempt to write the arguments of u, v, w, u', v' and w' over a common denominator OR to write the arguments in degrees ***(M1)***

$\frac{-9\pi}{12}, \frac{-4\pi}{12}, \frac{-\pi}{12}, \frac{4\pi}{12}, \frac{7\pi}{12}, \frac{12\pi}{12}$ OR $-135^\circ, -60^\circ, -15^\circ, 60^\circ, 105^\circ, 180^\circ$

THEN

arguments of u, v, w, u', v' and w' differ by $\frac{3\pi}{12}$ and $\frac{5\pi}{12}$ OR 45° and 75°

so arguments of polygon vertices differ by $\frac{\pi}{12}$ or 15° ***(A1)***

$n = 24$ ***A1***

METHOD 2

Let $z = r\,\text{cis}\,\theta \Rightarrow z^n = r^n \text{cis}(n\theta) = r^n \text{cis}(n\theta)$, where θ is the argument of u, v, w, u', v' and w'.

recognition to find $n\theta$ where $n = 6, 12, 18, \ldots$ and $\theta = -\frac{3\pi}{4}, -\frac{\pi}{3}, -\frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \pi$ ***(M1)***

when $n = 6 \Rightarrow (n\theta =) -\frac{9\pi}{2}, -2\pi, -\frac{\pi}{2}, 2\pi, \frac{7\pi}{2}, 6\pi$

when $n = 12 \Rightarrow (n\theta =) -9\pi, -4\pi, -\pi, 4\pi, 7\pi, 12\pi$ (which is not a multiple of 2π) ***(A1)***

$n = 24$ ***A1***

[3 marks]

Total [20 marks]